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The dependence of shoreline contaminant levels upon the siting of an effluent outfall

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It is shown that the advection–diffusion equation for contaminant dispersion in water of variable depth admits of an inverse approach in which the stochastic mean concentration at a fixed sensitive location is calculated for arbitrary discharge sites. A ray method is used to obtain simple approximate solutions for the inverse problem when the sensitive location is situated at the shoreline. This makes explicit the considerable improvements in shoreline pollution levels which can be gained by siting effluent outfalls further away from the shore.

1. Introduction

In this environmentally conscious age there is a heightened or even an exaggerated awareness of the consequences of siting an effluent outfall too close to a high-amenity area. Thus, it is becoming standard engineering practice to assess the consequences of choosing any particular discharge site (Yotsukura & Cobb 1972; Yotsukura & Sayre 1976; Fischer *et al.* 1979). While this does ensure that the pollution level is acceptable, it leaves to intuition the selection of the prospective discharge site and does not make explicit the improvements that can be gained by siting outfalls further from the shore. For example, on a uniformly sloping beach a one-third increase in the offshore distance reduces the peak concentration at the shoreline by a factor of two (Kay 1983).

One objective of the present paper is to give an inverse approach to effluent-disposal problems. We identify a sensitive location, such as a drinking-water intake at the side of a wide river, and we specify the continuous contaminant loading that is to be disposed of into the flow somewhere upstream of this location. We then ask what is the concentration level experienced at the sensitive location as a function of the discharge position?

The mathematical basis of the inverse approach is the use of Green's reciprocal theorem. To invoke this theorem we model the contaminant dispersion by means of a depth-averaged linear advection–diffusion equation, with the flow field independent of the discharge site. Thus there are several implicit restrictions upon the applicability of the inverse approach (the far-field approximation). In particular, stochastic or vertical variations in concentration are ignored, and the sensitive location is assumed to be well outside the zone (of order 100 m) in which momentum or buoyancy effects are important.

A second objective of this paper is to sharpen our intuition about the relative roles of the water depth, current strength and turbulence intensity in the selection of discharge sites. To do this a simple (but accurate) approximate method of solution is derived for the inverse problem when the sensitive location is situated at the shoreline, and some illustrative examples are studied in detail.

For a highly utilized shoreline, all locations can be thought of as being of equal importance, and the relevant consideration becomes the maximum concentration anywhere along the shoreline. A simplified version of this problem is addressed in §9 of this paper, with the depth profile modelled as having fixed offshore shape, but with a depthscale that changes along the flow. A quantitative example is given of how the presence of offshore sand banks shields the shoreline from the effects of discharges made at the far side of the sand banks.

2. Green's reciprocal theorem for advection–diffusion

For a vertically well-mixed flow the concentration $c_1(x, y)$ associated with a steady discharge of strength q at a point (x_1, y_1), satisfies the depth-averaged advection–diffusion equation

$$\nabla\cdot(h\boldsymbol{u}\,c_1)-\nabla\cdot(h\boldsymbol{\kappa}\cdot\nabla c_1) = q\delta(x-x_1)\,\delta(y-y_1), \tag{2.1a}$$

$$h(\boldsymbol{u}c_1-\boldsymbol{\kappa}\cdot\nabla c_1)\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\Omega, \tag{2.1b}$$

with

$$\nabla\cdot(h\boldsymbol{u}) = 0, \tag{2.2a}$$

$$h\boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\Omega. \tag{2.2b}$$

Here $h(x, y)$ is the water depth, $u(x, y)$ the steady flow velocity, $\boldsymbol{\kappa}$ the horizontal contaminant-dispersion tensor, δ the Dirac delta function, $\partial\Omega$ the boundary of the flow region, and $\boldsymbol{n}$ the outward normal.

The adjoint, reversed-flow, problem for a monitoring position (x_2, y_2) is

$$-\nabla\cdot(h\boldsymbol{u}c_2)-\nabla\cdot(h\boldsymbol{\kappa}\cdot\nabla c_2) = q\delta(x-x_2)\,\delta(y-y_2), \tag{2.3a}$$

$$h(\boldsymbol{u}c_2+\boldsymbol{\kappa}\cdot\nabla c_2) = 0 \quad \text{on } \partial\Omega. \tag{2.3b}$$

The best known of Green's theorems is the divergence theorem in two dimensions, which equates certain area and boundary integrals. The reciprocal theorem is a particular application of the divergence theorem to an adjoint pair of equations, such as (2.1a), (2.3a). We consider the identity

$$\int_\Omega c_2[\nabla\cdot(h\boldsymbol{u}c_1)-\nabla\cdot(h\boldsymbol{\kappa}\cdot\nabla c_1)]\,\mathrm{d}A+\int_\Omega c_1[\nabla\cdot(h\boldsymbol{u}c_2)+\nabla\cdot(h\boldsymbol{\kappa}\cdot\nabla c_2)]\,\mathrm{d}A$$

$$-\int_\Omega c_1\,c_2\,\nabla\cdot(h\boldsymbol{u})\,\mathrm{d}A$$

$$=\int_\Omega \nabla\cdot[c_2\,h(\boldsymbol{u}c_1-\boldsymbol{\kappa}\cdot\nabla c_1)]\,\mathrm{d}A+\int_\Omega \nabla\cdot[c_1\,h(\boldsymbol{u}c_2+\boldsymbol{\kappa}\cdot\nabla c_2)]\,\mathrm{d}A-\int_\Omega \nabla\cdot(c_1\,c_2\,h\boldsymbol{u})\,\mathrm{d}A$$

$$=\int_{\partial\Omega} c_2\,h(\boldsymbol{u}c_1-\boldsymbol{\kappa}\cdot\nabla c_1)\cdot\boldsymbol{n}\,\mathrm{d}l+\int_{\partial\Omega} c_1\,h(\boldsymbol{u}c_2+\boldsymbol{\kappa}\cdot\nabla c_2)\cdot\boldsymbol{n}\,\mathrm{d}l-\int_{\partial\Omega} c_1\,c_2\,h\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}l. \tag{2.4}$$

If c_1, c_2 are the solutions of (2.1a, b), (2.3a, b) and if the flow field satisfies (2.2a, b), then we can separately evaluate the area and boundary integrals to derive the result

$$q[c_2(x_1, y_1)-c_1(x_2, y_2)] = 0, \tag{2.5a}$$

i.e.

$$c_1(x_2, y_2) = c_2(x_1, y_1). \tag{2.5b}$$

Hence to calculate the concentration c_1 at the specified sensitive location (x_2, y_2) it suffices to solve the reversed-flow problem (2.3a, b) for the concentration c_2 at the prospective discharge site (x_1, y_1). The advantage of this adjoint problem is that, to compare the impacts of a whole range of alternative discharge sites, it is only necessary to evaluate the single function c_2 over the range of discharge positions. Because the flow is reversed, the 'influence' plume for c_2 spreads out in the opposite direction to that of the c_1 plume. In numerical schemes it would be essential to take account of this reversed orientation to avoid the computational catastrophe of solving a diffusion problem in the up-gradient direction.

3. Long-plume approximation

In a turbulent flow the dispersion tensor κ scales as the product of the water depth h and the friction velocity u_*:

$$\kappa_{11} \approx 5.9hu_*, \quad \kappa_{22} \approx 0.2hu_* \tag{3.1}$$

(Fischer *et al.* 1979, equations (4.46), (5.4)). In wide rivers the horizontal lengthscale B for topographic features greatly exceeds the water depth h, and the horizontal advection velocity u greatly exceeds u_*. Thus, in the advection–diffusion equations (2.1 a), (2.3 a) the dispersion tensor κ is numerically very small (i.e. very much smaller than Bu). This has the physical consequence that advection dominates diffusion, and the contaminant plume is extremely long and narrow (see figure 5.6 of Fischer *et al.* 1979).

Mathematically the elongation of the contaminant plume means that transverse gradients greatly exceed longitudinal gradients. To exploit this simplifying feature we follow Yotsukura & Cobb (1972) and use generalized coordinates (x, y) aligned along and across the flow (see figure 1). Thus, neglecting diffusion along the flow, we approximate the reversed-flow equations (2.3 a, b):

$$\partial_x(m_2 huc_2) + \partial_y\left(\frac{m_1}{m_2} h\kappa_{22}\,\partial_y c_2\right) = -q\delta(x-x_2)\,\delta(y-y_2), \tag{3.2a}$$

$$\frac{m_1}{m_2} h\kappa_{22}\,\partial_y c_2 = 0 \quad (y = 0), \tag{3.2b}$$

with

$$\partial_x(m_2 hu) = 0, \tag{3.3}$$

where $m_1\,\mathrm{d}x$, $m_2\,\mathrm{d}y$ are the incremental distances along and across the flow (Yotsukura & Sayre 1976, equation (2.1)).

4. Local solution (uniformly sloping beach)

Most human activities are confined to the land. Thus the sensitive location for water pollution can generally be expected to be at the shoreline. Sufficiently close to the shoreline the depth varies linearly with offshore distance:

$$h = h_0(y/y_0), \tag{4.1}$$

where h_0 is the water depth at a reference distance y_0. Also, close to the critical location the flow can be regarded as being locally straight and independent of x.

This idealization of a shoreline discharge has been studied by Kay (1983). In particular, if we model the longshore velocity, friction velocity, and the transverse diffusivity

$$u = u_0\left(\frac{h}{h_0}\right)^{\frac{1}{2}}, \quad u_* = u_{*0}\left(\frac{h}{h_0}\right)^{\frac{1}{2}}, \quad \kappa_{22} = \kappa_0\left(\frac{h}{h_0}\right)^{\frac{3}{2}}, \tag{4.2a, b, c}$$

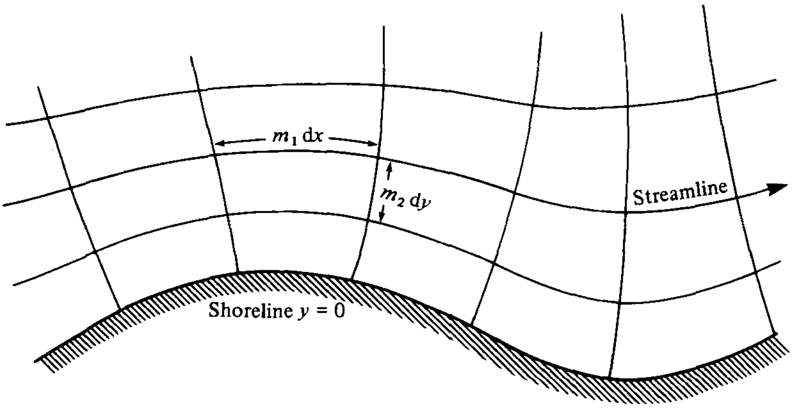

FIGURE 1. Sketch of the along- and cross-flow coordinate system.

then the solution for c_2 is

$$c_2 = \frac{4}{3\pi^{\frac{1}{2}}} \frac{q}{h_0 u_0 y_0} \left(\frac{u_0 y_0^2}{\kappa_0(x_2 - x)}\right)^{\frac{5}{2}} \exp\left\{-\frac{u_0 y_0 y}{\kappa_0(x_2 - x)}\right\} \quad (x < x_2). \tag{4.3)}$$

The modelling (4.2*a*–*c*) for the flow and turbulence follows from the assumptions that the pressure gradient driving the flow is independent of y, the eddy viscosity scales as hu_*, and there is proportionality between u and u_*. Kay (1983) also gives the solutions for other power-law models for the flow properties, and the more-complicated exact solutions for off-shore discharges.

Figure 2 shows the concentration at the sensitive location as a function of the discharge position. The chosen parameter values are

$$\kappa_0 = 0.2 h_0 u_{*0}, \quad u_{*0} = 0.1 u_0, \quad q = h_0 u_0 y_0. \tag{4.4}$$

If we regard the $c_2 = 1$ contour as demarcating the boundary between desirable and undesirable discharge sites (i.e. y_0 is the width of a well-mixed uniform channel necessary for the concentration to be just acceptable), then the proximity of the $c_2 = 0.2$ and $c_2 = 5$ contours indicates how easy it is either to achieve much-improved or totally unacceptable shoreline pollution levels.

5. Ray approximation

In the spirit of the WKB approximation we shall assume that, with a suitable choice for the amplitude and decay functions $a(x, y)$, $\phi(x, y)$, the local solution (4.3) can be extended:

$$c = a \exp(\pm\phi). \tag{5.1}$$

Here the + sign corresponds to the forward-flow problem, and the − sign to the reversed-flow problem. This variant of the WKB method is due to Cohen & Lewis (1967), and has been used by the author (Smith 1981) to study discharges well away from a shoreline.

If we substitute the ansatz (5.1) into the field equation (3.2*a*), and its forward-flow counterpart, then we generate terms proportional to $\exp(\pm\phi)$ and $\pm\exp(\pm\phi)$.

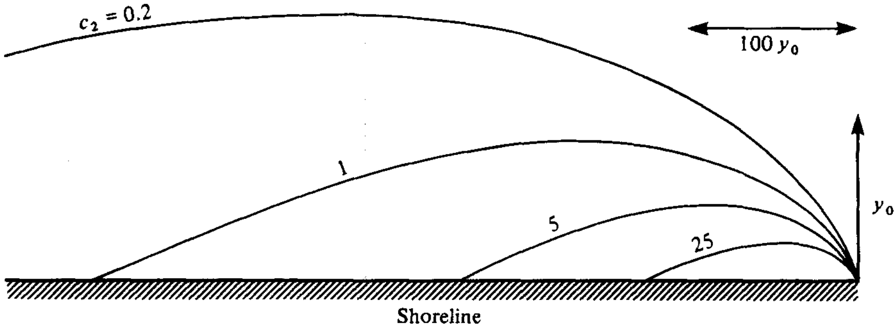

FIGURE 2. Kay's (1983) solution for the concentration experienced at the critical shoreline location as a function of the upstream discharge position. Note the 100:1 difference between the along- and cross-flow lengthscales.

Equating these groups of terms separately to zero, we have

$$m_2 hu\,\partial_x \phi - \frac{m_1}{m_2} h\kappa_{22}(\partial_y \phi)^2 = \partial_y\left(\frac{m_1}{m_2} h\kappa_{22}\,\partial_y a\right)\Big/ a, \tag{5.2}$$

$$m_2 hu\,\partial_x a - \partial_y\left(\frac{m_1}{m_2} h\kappa_{22}\, a\,\partial_y \phi\right) - \frac{m_1}{m_2} h\kappa_{22}\,\partial_y \phi\,\partial_y a = 0. \tag{5.3}$$

The essential features for the WKB-type approximation is that ϕ varies more rapidly than does the amplitude a. This is ensured by the occurrence of $1/\kappa_0$ in the exponential in (4.3). The corresponding approximation of (5.2) is

$$m_2 hu\,\partial_x \phi - \frac{m_1}{m_2} h\kappa_{22}(\partial_y \phi)^2 = 0. \tag{5.4}$$

The error incurred is of second order in the numerical smallness of the diffusivity κ_{22} relative to typical horizontal length and advection-velocity scales (i.e. an error of order $(h/B)^2\,(u_*/u)^2$).

Close to the critical shoreline location $(x_2, 0)$ the depth becomes linear, and for compatibility with the location solution (4.3) we must have

$$\phi \sim \frac{y}{x_2 - x} \lim_{y\to 0} \frac{uym_2^2}{m_1\,\kappa_{22}}, \tag{5.5}$$

$$a \sim \frac{4q}{3\pi^{\frac{1}{2}}(x_2-x)^{\frac{5}{2}}} \lim_{y\to 0} \frac{m_2^4\,y^4 u^{\frac{3}{2}}}{m_1^{\frac{5}{2}}\,h\kappa_{22}^{\frac{5}{2}}}. \tag{5.6}$$

The limiting process amounts to taking the reference position y_0 sufficiently close to shore so that, despite the varying geometry, the representations (4.1), (4.2) are locally valid.

What has been achieved with the ray approximation is that the parabolic equation (3.1 a) has been replaced by the hyperbolic equations (5.3), (5.4). Thus instead of the concentration at any one point being influenced by all other points, the influence only comes outwards from the source (along rays). This permits (5.3), (5.4) to be solved by marching methods. The main limitation of the ray approximation is that it is only applicable for wide rivers or open coastlines. It fails to account for the reflection at any far shoreline.

6. Decay exponent

If we introduce the diffusivity/velocity ratio

$$F = \frac{m_1 \kappa_{22}}{m_2^2 u} \approx 0.2h \frac{m_1}{m_2^2} \frac{u_*}{u}, \tag{6.1}$$

then the eikonal equation (5.4) for ϕ can be rewritten

$$\partial_x \phi - F(\partial_y \phi)^2 = 0. \tag{6.2}$$

This first-order partial differential equation can be solved by the method of characteristics or rays (Courant & Hilbert 1962, §2). For the reversed-flow problem the ray direction is along the vector

$$(-1, \Delta) \quad \text{with} \quad \Delta = 2F \partial_y \phi \tag{6.3}$$

(i.e. in the upstream direction). We introduce a ray parameter s so that differentiation along a ray is defined:

$$\frac{\partial}{\partial s} = -\frac{\partial}{\partial x} + \Delta \frac{\partial}{\partial y}. \tag{6.4}$$

In particular, to extend a ray we need to solve the ray-tracing equations

$$\frac{\partial x}{\partial s} = -1, \quad \frac{\partial y}{\partial s} = \Delta. \tag{6.5a, b}$$

The powerful feature of the method of characteristics is that not only do x, y satisfy elementary ordinary differential equations along rays, but so also do the decay exponent ϕ and the ray orientation Δ:

$$\frac{\partial \phi}{\partial s} = \frac{\Delta^2}{4F}, \tag{6.6a}$$

$$\frac{\partial \Delta}{\partial s} = \Delta^2 \frac{\partial_y F}{2F} - \Delta \frac{\partial_x F}{F}, \tag{6.6b}$$

where the diffusivity/velocity ratio F and its spatial derivations are evaluated at the present location of the ray.

Over a wide range of roughness the ratio u_*/u lies in the narrow band 0.05–0.1. Thus, essentially F is proportional to the local water depth, and increases linearly near the shoreline. From (6.5b, 6.6b) we can infer that the rays emerge from the critical shoreline location $(x_2, 0)$ with the orientation initially zero and increasing linearly with s (see figure 3). This means that the appropriate initial conditions for the ray-tracing equations (6.5), (6.6) are

$$x = x_2, \quad y = 0, \quad \phi = 0, \quad \Delta = 0, \quad \frac{\partial \Delta}{\partial s} = p \quad \text{at} \quad s = 0, \tag{6.7}$$

where the initial curvature p serves to label the individual rays.

The subsequent orientation of the rays depends upon the local behaviour of the diffusivity/velocity ratio $F(x, y)$, as can be seen in (6.6b). If the water depth increases upstream ($\partial_x F < 0$), or if the transverse depth slope increases with y, then the rate of divergence of the rays is increased. Shallow water upstream, or a flattening out of the water depth, has the opposite effect of reducing the rate of divergence of the rays. Thus we can infer from (6.6a) that ϕ will be larger when the upstream discharge is in relatively deep water. Equivalently, a deep-water discharge leads to lower concentrations c_2 experienced at the critical location (see §8 below).

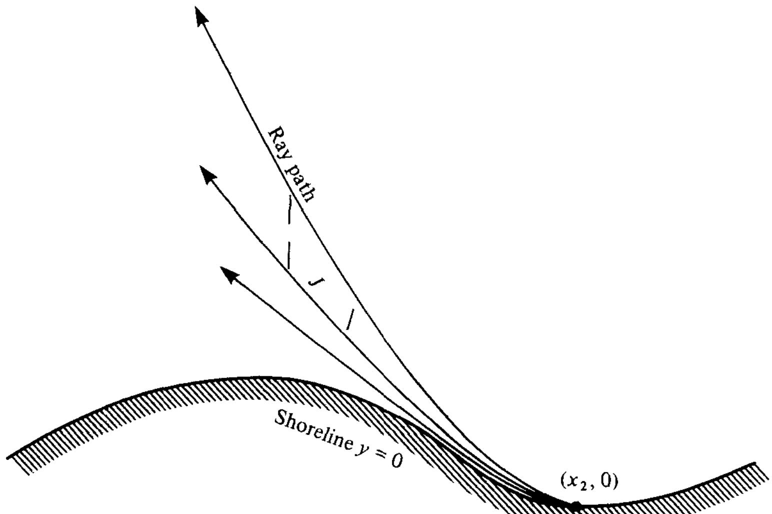

FIGURE 3. Sketch of backwards-going rays emerging from the critical shoreline location.

7. Amplitude factor

The derivatives of the amplitude factor $a(x,y)$ in the transport equation (5.3) exactly conform to the ray direction. Thus, making use of the definitions (6.3), (6.4), we can rewrite the transport equation:

$$\frac{1}{a}\frac{\partial a}{\partial s}+\frac{1}{2}\frac{\partial \Delta}{\partial y}+\tfrac{1}{2}\Delta\frac{\partial}{\partial y}(m_2 hu)\frac{1}{m_2 hu} = 0. \tag{7.1}$$

Differentiating Δ with respect to y involves moving across the rays. This leads us to consider the area J between adjacent rays:

$$J = \frac{\partial x}{\partial p}\frac{\partial y}{\partial s}-\frac{\partial x}{\partial s}\frac{\partial y}{\partial p}. \tag{7.2}$$

Using the chain rule, relating x-, y- and p-, s-derivatives, we can obtain the general result

$$\frac{\partial J}{\partial s} = J\frac{\partial}{\partial x}\left(\frac{\partial x}{\partial s}\right)+J\frac{\partial}{\partial y}\left(\frac{\partial y}{\partial s}\right). \tag{7.3}$$

For the case under consideration here we have (from (6.5a, b))

$$\frac{\partial J}{\partial s} = J\frac{\partial \Delta}{\partial y}. \tag{7.4}$$

This result (7.4), together with the mass-conservation equation (3.2), permits us to rewrite the transport equation (7.1):

$$\frac{1}{a}\frac{\partial a}{\partial s}+\frac{1}{2}\frac{1}{J}\frac{\partial J}{\partial s}+\frac{1}{2}\frac{1}{m_2 hu}\frac{\partial}{\partial s}(m_2 hu) = 0. \tag{7.5}$$

Integration with respect to s yields the result

$$aJ^{\frac{1}{2}}(m_2 hu)^{\frac{1}{2}} = \text{constant along rays}. \tag{7.6}$$

Thus greatest amplitudes are to be found where the rays are closest together or where the water depth is least. Of course, to draw conclusions about the concentrations

$$c_2 = a\exp(-\phi), \tag{7.7}$$

it is also necessary to allow for the exponential decay factor.

Near the critical shoreline location we have, using (6.5), (6.7), (7.2),

$$x = x_2 - s, \quad y = \tfrac{1}{2}ps^2, \quad J = \tfrac{1}{2}s^2. \tag{7.8}$$

Thus, to satisfy the local asymptote (5.6), the ray constant must be given by

$$\begin{aligned}\text{constant} &= \frac{2^{\frac{3}{4}}q}{3\pi^{\frac{1}{2}}}\lim_{y\to 0}\left(\frac{m_2^{\frac{9}{2}}y^{\frac{13}{4}}u^2}{m_1^{\frac{5}{4}}h^{\frac{1}{2}}\kappa_{22}^{\frac{5}{2}}}\right)p^{\frac{3}{4}} \\ &= q(2p)^{\frac{3}{4}}\Big/3\pi^{\frac{1}{2}}\left[\lim_{y\to 0}\frac{F}{y}\right]^{\frac{5}{2}}\left[\lim_{y\to 0}\frac{m_2hu}{y^{\frac{3}{2}}}\right]^{\frac{1}{2}},\end{aligned} \tag{7.9}$$

where it is implicit that we are also taking the limit $x\to x_2$.

8. Separation-of-variables solution

For any real topography the first (and hardest) task would be to measure or to calculate the velocity field, and inevitably the subsequent ray calculations would have to be numerical. With the continuing objective of sharpening our intuition, we seek instead an idealized class of problems which is amenable to exact solution. Specifically, we taken the diffusivity/velocity ratio F to have a separation-of-variables form

$$F = \frac{\kappa_0}{u_0}f_1(x)f_2(y). \tag{8.1}$$

This corresponds to the depth profile having a fixed offshore shape $f_2(y)$, but with a depthscale that varies along the flow as $f_1(x)$.

The solution of the ray equations (6.5), (6.6) is given by

$$x = x_2 - s, \quad [f_2'(0)]^{\frac{1}{2}}\int_0^y f_2(Y)^{-\frac{1}{2}}\,\mathrm{d}Y = \frac{(2p)^{\frac{1}{2}}\int_x^{x_2} f_1(X)\,\mathrm{d}X}{f_2(x_2)}, \tag{8.2}$$

and the corresponding solution for the decay exponent ϕ is

$$\phi = \frac{p\dfrac{u_0}{\kappa_0}\displaystyle\int_x^{x_2} f_1(X)\,\mathrm{d}X}{2f_1(x_2)^2 f_2'(0)}. \tag{8.3}$$

Eliminating p in favour of x, y, we have the explicit solution for ϕ:

$$\phi = \frac{u_0}{4\kappa_0}\frac{\left[\displaystyle\int_0^y f_2(Y)^{-\frac{1}{2}}\,\mathrm{d}Y\right]^2}{\displaystyle\int_x^{x_2} f_1(X)\,\mathrm{d}X}. \tag{8.4}$$

Evaluating the Jacobian (7.2), we have

$$J = \frac{\displaystyle\int_x^{x_2} f_1(X)\,\mathrm{d}X f_2(y)^{\frac{1}{2}}}{(2p)^{\frac{1}{2}}f_1(x_2)f_2'(0)^{\frac{1}{2}}}. \tag{8.5}$$

Using (7.6), (7.9), (8.2), this leads to a solution for the amplitude factor

$$a = \frac{\frac{1}{3}q\pi^{-\frac{1}{2}}\left[\int_0^y f_2(Y)^{-\frac{1}{2}}\,\mathrm{d}Y\right]^2}{(m_2 hu)^{\frac{1}{2}}\left[\lim_{y\to 0}\frac{m_2 hu}{y^{\frac{3}{2}}}\right]^{\frac{1}{2}}\left[\frac{\kappa_0}{u_0}\int_x^{x_2} f_1(X)\,\mathrm{d}X\right]^{\frac{5}{2}} f_2(y)^{\frac{1}{4}} f_2'(0)^{\frac{5}{4}}}. \tag{8.6}$$

It is of interest to note some general features of the solutions (8.4), (8.6). The denominator in the expression for ϕ gives greatest weight to sections of shoreline with the steepest beach slopes (i.e. with $f_1(x)$ large). Thus the width of the reversed-flow contaminant plume is determined primarily by the regions of steep slope. In the same way, but exaggerated by a $\frac{5}{2}$ power, the amplitude factor $a(x, y)$ is diminished most rapidly in regions of large $f_1(x)$.

The behaviour as regards the y-dependence is totally reversed. It is regions of small $f_2(y)$ (sandbanks) which dominate the expressions (8.4), (8.6). The explanation for this difference is that in the cross-flow direction, a region of low diffusivity is an effective barrier to pollution. However, in the flow direction there is no such blockage because the contaminant is carried by the flow past the region of low diffusivity.

As a simple extension of Kay's solution (4.3), we take $m_1 = 1$ and we model the longshore velocity, depth and diffusivity by

$$u = u_0\left(\frac{y}{y_0}\right)^{\frac{1}{2}}\Big/ f_1(x) m_2^3, \quad h = h_0 \frac{y}{y_0} f_1(x) m_2^2, \quad \kappa_{22} = \kappa_0 \left(\frac{y}{y_0}\right)^{\frac{3}{2}} m_2, \tag{8.7a, b, c}$$

i.e.

$$f_2(y) = \frac{y}{y_0}, \tag{8.7d}$$

where $f_1(x) m_2(x)$ measures the relative steepness of the beach at different points along the shoreline. The resulting formula for the concentration distribution is

$$c_2 = \frac{4}{3\pi^{\frac{1}{2}}} \frac{q}{h_0 u_0 y_0} \left[\frac{u_0 y_0^2}{\kappa_0 \int_x^{x_2} f_1(X)\,\mathrm{d}X}\right]^{\frac{5}{2}} \exp\left\{\frac{-u_0 y_0 y}{\kappa_0} \int_x^{x_2} f_1(X)\,\mathrm{d}X\right\}. \tag{8.8}$$

We remark that this is an exact solution of (3.2a) rather than a first-order approximation.

Figures 4(a, b) show the contrasting cases

$$f_1 = \exp\left(\frac{x}{500 y_0}\right), \quad f_1 = \exp\left(\frac{-x}{500 y_0}\right), \tag{8.9}$$

with κ_0 and q specified as in figure 1 (i.e. by (4.4)). The vulnerability of the critical location to upstream discharges is greatest when the water is shallower upstream. In both cases the benefits of moving the discharge well away from the shoreline are just as marked as in the constant-slope case (figure 2).

9. Maximum shoreline concentration

Throughout the above analysis we have emphasized the dependence of the decay exponent and the amplitude factor upon the discharge position. However, as can be seen in the explicit solutions (8.4), (8.6), there is also a dependence upon the critical shoreline location x_2. As noted in §1, for highly utilized shorelines all locations are of equal importance, and we are concerned to evaluate the maximum shoreline

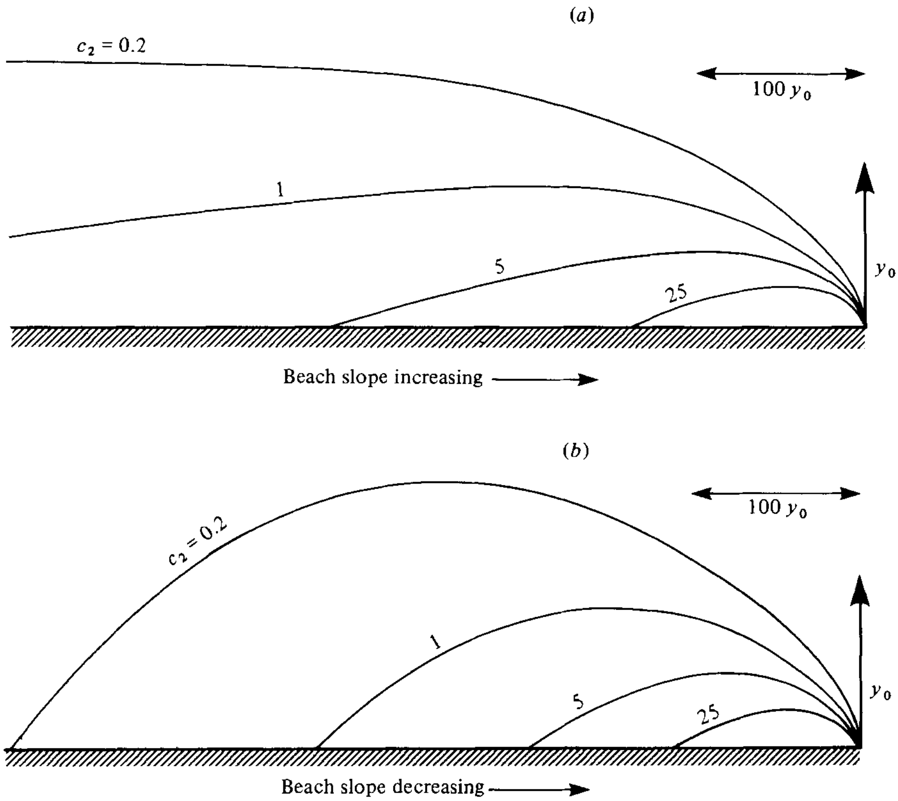

FIGURE 4. Concentration experienced at the critical shoreline location as a function of the upstream discharge position when the beach slope is (a) exponentially increasing downstream, (b) exponentially decreasing downstream, with an e-folding distance of $500y_0$.

concentration. From the ray ansatz (5.1), we infer that this worst case arises when

$$\frac{1}{a}\frac{\partial a}{\partial x_2} = \frac{\partial \phi}{\partial x_2}. \tag{9.1}$$

For the idealized geometrics considered in §8, (9.1) implicitly determines x_2 for a given discharge position (x, y):

$$\int_x^{x_2} f_1(X)\,\mathrm{d}X = \frac{1}{10}\frac{u_0}{\kappa_0}\left[\int_0^y f_2(Y)^{-\frac{1}{2}}\,\mathrm{d}Y\right]^2. \tag{9.2}$$

Thus, for nearshore discharge positions, the downstream distance for maximum concentration varies linearly with the offshore distance of the discharge (Kay 1983). The proportionality factor involves $1/\kappa_0$, allowing for the considerable elongation of the contaminant plume.

Substituting the above result (9.2) for x_2 into (8.4), (8.6), we find that the peak shoreline concentration is given by

$$c_{\max} = \frac{q 10^{\frac{5}{2}} \mathrm{e}^{-\frac{5}{2}}}{3\pi^{\frac{1}{2}}}\left\{(m_2 hu)^{\frac{1}{2}}\left[\lim_{y\to 0}\frac{m_2 hu}{y^{\frac{3}{2}}}\right]^{\frac{1}{2}} f_2'(0)^{\frac{5}{4}} f_2(y)^{\frac{1}{4}}\left[\int_0^y f_2(Y)^{-\frac{1}{2}}\,\mathrm{d}Y\right]^3\right\}^{-1}. \tag{9.3}$$

Again, for nearshore discharges we recover Kay's (1982) uniform-beach result that the peak concentration at the shoreline is inversely proportional to the $\frac{5}{2}$ power of

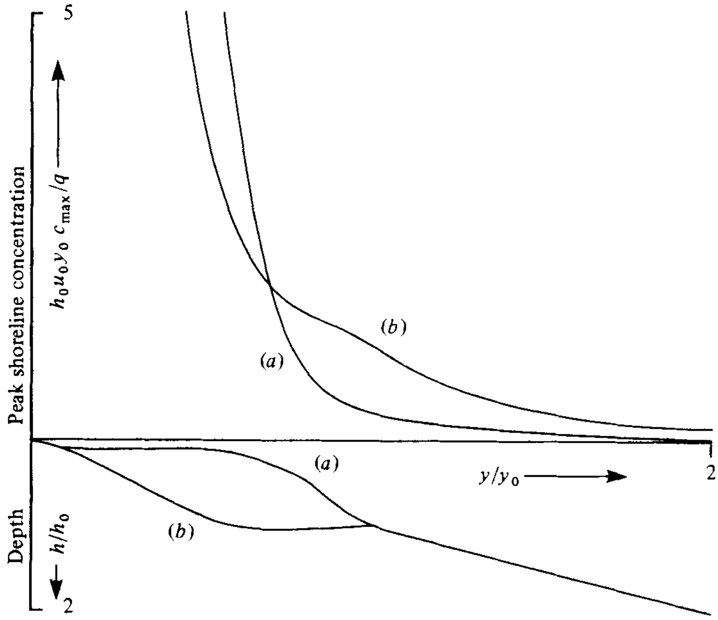

FIGURE 5. The peak concentration at the shoreline as a function of discharge distance away from the shoreline for depth profiles (*a*) with, and (*b*) without, offshore sandbanks.

the offshore distance of the discharge. Remarkably, this formula does not involve x. In deeper water the more-rapid widening of the contaminant plume is exactly counterbalanced by the earlier occurrence of the shoreline maximum.

The comparatively large power of the integral term in the denominator of (9.3) means that it is the full-depth topography that is important, and not just the depth at the discharge. For example, a region of shallow water (i.e. of small f_2) inshore of the discharge can profoundly reduce the shoreline concentration. Physically what happens is that the region of low transverse diffusivity delays the transport of contaminant towards the shoreline, while diffusion away from shore continues to reduce the concentration. As a quantitative example, figure 5 shows the peak shoreline concentrations for the two depth profiles

$$h = \frac{h_0 y}{y_0}\left[1-2\exp\left(\frac{-y_0^2}{4y(y_0-y)}\right)\right] \quad (0 < y < y_0), \tag{9.4a}$$

$$h = \frac{h_0 y}{y_0}\left[1+2\exp\left(\frac{-y_0^2}{4y(y_0-y)}\right)\right] \quad (0 < y < y_0), \tag{9.4b}$$

both with

$$h = \frac{h_0 y}{y_0} \quad (y_0 < y);$$

κ_0, u_{*0}, q are as specified in (4.4). For nearshore discharges, the more-efficient diffusion and advective flushing in the deep-water case (9.4*b*) means that the shoreline concentrations are lower than for the shallow-water case (9.4*a*). However, if the pipeline can be extended beyond the sand banks, then advantage can be taken of the shielding effect of the shallow water. Indeed, the steepness of the concentration profile (*a*) in figure 5 shows that spectacular improvements in shoreline pollution levels can be obtained for quite modest relocations of the discharge site – as much as a halving of the peak shoreline concentration for a one-sixth increase in the offshore distance of the effluent outfall.

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